As the airflow velocity reaches $U = 13$ m/s, the wide region of LCO ceases to exist and gives way to a static response.

As one would probably expect with further increase of the angle of attack, the static response takes over completely, as seen in Fig. 2d for $\alpha_0 = 4\delta$.

Often in nonlinear dynamics it is of interest to study the effects of changes in initial conditions on the system's response. A variety of different initial conditions were tested. The results show that there are some subtle changes in the response of the sytem. For example, an angle of attack of $\alpha_0 = 3\delta$ was considered, and the results were obtained for three sets of initial conditions: $\boldsymbol{x}_1(0) = [0, 2\delta, 0]^T$ (as before), $\boldsymbol{x}_2(0) = [2\delta, 0, 0]^T$, and $\boldsymbol{x}_3(0) = [0, 0, 0.5]^T$. Note that $\dot{\boldsymbol{x}}(0) = [0, 0, 0]^T$ in all three sets. For most of the $\boldsymbol{U}$ interval, there are no substantial differences due to different initial conditions. By this we mean that, for all considered initial conditions, regions of stable LCO, or chaotic motions, or static points begin and end at the same values of the airflow velocity $\boldsymbol{U}$. Also, the numerical values of the response for static points and LCO remain unchanged at least to the second digit after the decimal point, whereas values of the averaged chaotic response may change in the first or second digits after the point. However, in the interval of $\boldsymbol{U} \cong 7$–13 m/s, different initial disturbances lead to the different behavior patterns of the system: The first set of initial conditions produces two small regions of LCO at $\boldsymbol{U} = 8.5$–9.5 and 11.5–12.5 m/s, and the second set results in the dominance of static points, and at the same time, the third set gives rise to an LCO region for $\boldsymbol{U} = 9$–12.8 m/s. For brevity, these results are not shown graphically here, but they are available from the authors.

Conclusions

Using a standard state-space approximation to Theodorsen aerodynamics in a model developed in Refs. 1 and 2, the behavior of a three-degree-of-freedom typical airfoil section has been studied via numerical time integration, with the focus on the effects of the mean or steady angle of attack. A rich variety of nonlinear behavior has been observed; the main tendency displayed by the system as the angle of attack increases was the transition from the predominance of LCOs to a static deflection.

Note, finally, that the magnitude of the flap motion is of the order of the freeplay gap δ. For a typical airfoil, δ is of 1 deg or less. Similarly, the pitch motion is of the order of the freeplay gap δ, and the plunge motion is of the order of the freeplay gap times airfoil chord. Hence, the motions involved are sufficiently small that classical linear aerodynamic theory is still valid, and thus we have used Theodorsen's theory in the present Note. The numerical accuracy of the results shown is well within the dimensions of the symbols plotted in the two figures.

References

[1]Conner, M. D., "Nonlinear Aeroelasticity of an Airfoil Section with Control Surface Freeplay," Ph.D. Dissertation, Dept. of Mechanical Engineering and Material Science, Duke Univ., Durham, NC, 1996.

[2]Conner, M. D., Tang, D. M., Dowell, E. H., and Virgin, L. N., "Nonlinear Behavior of a Typical Airfoil Section with Control Surface Freeplay: One Numerical and Experimental Study," *Journal of Fluids and Structures*, Vol. 11, No. 1, 1997, pp. 89–109.

[3]Tang, D. M., Conner, M. D., and Dowell, E. H., "Reduced-Order Aerodynamic Model and Its Application to a Nonlinear Aeroelastic System," *Journal of Aircraft*, Vol. 35, No. 2, 1998, pp. 332–338.

[4]Tang, D. M., Dowell, E. H., and Virgin, L. N., "Limit Cycle Behavior of an Airfoil with a Control Surface," *Journal of Fluids and Structures*, Vol. 12, No. 7, 1998, pp. 839–858.

[5]Edwards, J. W., Ashley, H., and Breakwell, J. V., "Unsteady Aerodynamic Modeling for Arbitrary Motions," *AIAA Journal*, Vol. 17, No. 4, 1979, pp. 365–374.

[6]Henon, M., "On the Numerical Computation of Poincaré Maps," *Physica D*, Vol. 5D, Nos. 2, 3, 1982, pp. 412–414.

[7]Theodorsen, T., "General Theory of Aerodynamic Instability and Mechanism of Flutter," NACA Rept. 496, 1935.

A. Plotkin
Associate Editor

Further Research for Sensitivity Analyses of Discrete Periodic Systems

Ying Gu,* Lifen Chen,† and Wenliang Wang‡
Fudan University,
200433 Shanghai, People's Republic of China

Introduction

EIGENDERIVATIVES are extremely useful for determining the sensitivities of dynamic responses to system parameter variations. A wide class of physical systems can be approximated by systems of ordinary linear differential equations with periodic coefficients. Unlike the systems with constant coefficients whose sensitivity analysis techniques are well known, the sensitivity analyses of these periodic coefficient equations are somewhat less familiar. In general, the methods developed for the constant systems are not applicable for periodic coefficient systems because the closed form of a Floquet transition matrix is generally unavailable. Usually the finite difference approach is used to calculate the sensitivities. This approach is easy to implement but costly because of heavy computation time. Also, a proper step size is sometimes difficult to determine. Lim and Chopra[1] employed a chain rule differentiation approach to obtain the derivatives of a Floquet transition matrix $\boldsymbol{Q}$ with respect to parameters, which is more efficient than the finite difference approach for calculating the derivatives of the eigenexponents. Later, Lu and Murthy[2] introduced the direct analytical approach, and it made the sensitivity analyses of periodic coefficient systems much more effective. This Note extends the method of Lu and Murthy to the cases in which the period T can be dependent on the system parameters.

Theory of Periodic Coefficient Systems

A general homogeneous, first-order periodic coefficient system is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{1}$$

where $\boldsymbol{A}(t)$ is an $n \times n$ periodic matrix, with period T, and $\boldsymbol{x}(t)$ is a column vector of state variables. One set of the solutions of Eq. (1) can be of the form

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0) \tag{2}$$

where $\boldsymbol{\Phi}(t)$ is a standard fundamental solution matrix or state transition matrix. The Floquet theorem[3] states that $\boldsymbol{\Phi}(t)$ can be of the form

$$\boldsymbol{\Phi}(t) = \boldsymbol{P}(t)e^{\boldsymbol{K}t}, \qquad \boldsymbol{P}(0) = \boldsymbol{I} \tag{3}$$

where $\boldsymbol{P}(t)$ is an invertible periodic matrix of order n, with period T, whereas $\boldsymbol{K}$ is a steady $n \times n$ matrix that can be parameter dependent. When $t = T$, we have

$$\boldsymbol{x}(T) = \boldsymbol{\Phi}(T)\boldsymbol{x}(0) \tag{4}$$

From Eq. (3), we obtain

$$\boldsymbol{x}(T) = e^{\boldsymbol{K}T}\boldsymbol{x}(0) \tag{5}$$

Hence,

$$\boldsymbol{\Phi}(T) = e^{\boldsymbol{K}T} \tag{6}$$

Definition: If $\boldsymbol{\Phi}(T)$ and $\boldsymbol{K}$ are the Floquet transition matrix and the steady matrix of a periodic system, then their eigenvalues are said to be multipliers and eigenexponents of the system, respectively.

Received Aug. 20, 1998; revision received Dec. 30, 1998; accepted for publication Dec. 31, 1998.

*Doctoral Candidate, Department of Applied Mechanics. E-mail: ygudx@public6.sta.net.cn.

†Lecturer, Department of Applied Mechanics.

‡Professor, Department of Applied Mechanics.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenexponents of the system and $\rho_1, \rho_2, \ldots, \rho_n$ be the multipliers of the same system. Then $\rho_k = e^{\lambda_k T}$, $k = 1, \ldots, n$.

Hence,

$$\lambda_k = (1/T)\ln \rho_k = \alpha_k + i\omega_k, \qquad k = 1, \ldots, n \tag{7a}$$

From which the real and imaginary parts of λ_k can be given by

$$\alpha_k = (1/2T)\ln\left[(\rho_k)_R^2 + (\rho_k)_I^2\right]$$
$$\omega_k = \frac{1}{T}\tan^{-1}\left[\frac{(\rho_k)_I}{(\rho_k)_R}\right] \tag{8}$$

Because $\tan^{-1}$ is multivalued, ω_k can be obtained only as a basic frequency ω_k^* plus or minus any integer multiple of $2\pi/T$; hence,

$$\lambda_k = \lambda_k^* + i(2m\pi/T) \tag{7b}$$

where the principal eigenexponent $\lambda_k^* = \alpha_k + i\omega_k^*$. As in Refs. 1 and 2, suppose that the latent roots of $\boldsymbol{K}$ are distinct. Then there exist complete biorthonormal left and right eigenvectors sets

$$\boldsymbol{L} = \left\lfloor \boldsymbol{l}_{(1)} \vdots \boldsymbol{l}_{(2)} \vdots \cdots \vdots \boldsymbol{l}_{(n)} \right\rfloor, \qquad \boldsymbol{R} = \left\lfloor \boldsymbol{r}_{(1)} \vdots \boldsymbol{r}_{(2)} \vdots \cdots \vdots \boldsymbol{r}_{(n)} \right\rfloor$$

such that

$$(\boldsymbol{K} - \lambda_k \boldsymbol{I})\boldsymbol{r}_{(k)} = \boldsymbol{0}, \qquad \boldsymbol{l}_{(j)}^T(\boldsymbol{K} - \lambda_j \boldsymbol{I}) = \boldsymbol{0}, \qquad j, k = 1, \ldots, n \tag{9a}$$

$$\boldsymbol{L}^T\boldsymbol{R} = \boldsymbol{I}, \qquad \boldsymbol{L}^T\boldsymbol{K}\boldsymbol{R} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Lambda \tag{9b}$$

From Eq. (9b), we obtain

$$\boldsymbol{L}^T e^{\boldsymbol{K}t}\boldsymbol{R} = e^{\boldsymbol{L}^T\boldsymbol{K}t\boldsymbol{R}} = e^{\Lambda t} \tag{10}$$

which can be written, using Eq. (8), as

$$\boldsymbol{L}^T e^{\boldsymbol{K}t}\boldsymbol{R} = \operatorname*{diag}_k\{\exp(\alpha_k t)[\cos(\omega_k t) + i\sin(\omega_k t)]\} \tag{11}$$

Then one has another form of the solution set:

$$\boldsymbol{x}(t) = \boldsymbol{P}(t)\boldsymbol{R}\operatorname*{diag}_k\{\exp(\alpha_k t)[\cos(\omega_k t) + i\sin(\omega_k t)]\}\boldsymbol{L}^T\boldsymbol{x}(0) \tag{12a}$$

Introducing the periodic modal matrix[2]

$$\boldsymbol{U}(t) = \boldsymbol{P}(t)\boldsymbol{R} \tag{13}$$

then

$$\boldsymbol{x}(t) = \boldsymbol{U}(t)\operatorname*{diag}_k\{\exp(\alpha_k t)[\cos(\omega_k t) + i\sin(\omega_k t)]\}\boldsymbol{L}^T\boldsymbol{x}(0) \tag{12b}$$

It is clear that the stability of the trivial solution of a periodic system is determined by α_k. The imaginary part ω_k represents the frequency. Reference 4 states that part of the periodicity of the motion is in the root ω_k and part in the periodic function $\boldsymbol{U}(t)$, which adjusts itself accordingly.

Eigenvalue Derivatives

Substituting Eq. (12b) into Eq. (1) and eliminating the nonzero factor,

$$\operatorname*{diag}_k\{\exp(\alpha_k t)[\cos(\omega_k t) + i\sin(\omega_k t)]\}\boldsymbol{L}^T\boldsymbol{x}(0)$$

the governing equation of the periodic modal matrix $\boldsymbol{U}(t)$ is derived:

$$\dot{\boldsymbol{U}}(t) = \boldsymbol{A}(t)\boldsymbol{U}(t) - \boldsymbol{U}(t)\Lambda \tag{14}$$

Differentiating Eq. (14) with respect to design parameter p yields

$$\dot{\boldsymbol{U}}_{,p}(t) = \boldsymbol{A}_{,p}(t)\boldsymbol{U}(t) + \boldsymbol{A}(t)\boldsymbol{U}_{,p}(t) - \boldsymbol{U}_{,p}(t)\Lambda - \boldsymbol{U}(t)\Lambda_{,p} \tag{15}$$

It is convenient that another nonsingular periodic matrix $\boldsymbol{V}(t)$ is introduced such that

$$\boldsymbol{V}^T(t)\boldsymbol{U}(t) = \boldsymbol{I} \tag{16}$$

Premultiplying Eq. (15) by $\boldsymbol{V}^T$ and differentiating Eq. (16) with respect to time, we have

$$\boldsymbol{V}^T(t)\dot{\boldsymbol{U}}_{,p}(t) = \boldsymbol{V}(t)^T\boldsymbol{A}_{,p}(t)\boldsymbol{U}(t) + \boldsymbol{V}(t)^T\boldsymbol{A}(t)\boldsymbol{U}_{,p}(t) - \boldsymbol{V}(t)^T\boldsymbol{U}_{,p}(t)\Lambda - \Lambda_{,p} \tag{17}$$

$$\dot{\boldsymbol{V}}^T(t)\boldsymbol{U}(t) + \boldsymbol{V}^T(t)\dot{\boldsymbol{U}}(t) = \boldsymbol{0} \tag{18}$$

Utilizing Eqs. (14) and (18), we obtain

$$\dot{\boldsymbol{V}}^T(t) = -\boldsymbol{V}^T(t)\boldsymbol{A}(t) + \Lambda\boldsymbol{V}^T(t) \tag{19}$$

Right multiplying Eq. (19) by $\boldsymbol{U}_{,p}(t)$ gives

$$\dot{\boldsymbol{V}}^T(t)\boldsymbol{U}_{,p}(t) = -\boldsymbol{V}^T(t)\boldsymbol{A}(t)\boldsymbol{U}_{,p}(t) + \Lambda\boldsymbol{V}^T(t)\boldsymbol{U}_{,p}(t) \tag{20}$$

Equation (17) added to Eq. (20) makes

$$\dot{\boldsymbol{H}} = \boldsymbol{V}^T(t)\boldsymbol{A}_{,p}(t)\boldsymbol{U}(t) + (\Lambda\boldsymbol{H} - \boldsymbol{H}\Lambda) - \Lambda_{,p} \tag{21a}$$

or

$$\dot{h}_{jk} = \boldsymbol{v}_j^T\boldsymbol{A}_{,p}\boldsymbol{u}_k + (\lambda_j - \lambda_k)h_{jk} - \lambda_{k,p}\delta_{jk}, \qquad j, k = 1, \ldots, n \tag{21b}$$

where

$$\boldsymbol{H}[h_{jk}] = \boldsymbol{V}^T(t)\boldsymbol{U}_{,p}(t) \tag{22}$$

If the matrix $\boldsymbol{H}$ has been found, then the sensitivity of the periodic modal matrix can be acquired:

$$\boldsymbol{U}_{,p}(t) = \boldsymbol{U}(t)\boldsymbol{H}(t) \tag{23}$$

Letting $j = k$ in Eq. (21b) gives

$$\lambda_{k,p} = \boldsymbol{v}_k^T\boldsymbol{A}_{,p}\boldsymbol{u}_k - \dot{h}_{kk}, \qquad k = 1, \ldots, n \tag{24}$$

Integrating Eq. (24) over one period T and noting that eigenexponents are independent of time, then the sensitivity of eigenexponent λ_k can be found as follows:

$$\lambda_{k,p} = \frac{1}{T}\int_0^T \boldsymbol{v}_k^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}t - \frac{1}{T}[h_{kk}(T) - h_{kk}(0)], \qquad k = 1, \ldots, n \tag{25}$$

where $\boldsymbol{H}[h_{jk}]$ is equivalent to the matrix $[\partial\eta_{jk}(t)/\partial p]$ in Ref. 2. The authors of Ref. 2 considered that $[\partial\eta_{jk}(t)/\partial p]$ must be a periodic function of time; however, this is not always true.

Key theorem: For a general periodic function matrix $\boldsymbol{G}(t, p)$, with period $T(p)$, the derivative with respect to design parameter p is of the form

$$\boldsymbol{G}_{,p}(t) = \dot{\boldsymbol{G}}(t)T_{,p} + \boldsymbol{G}_{,p}(t+T), \qquad T_{,p} \equiv \frac{\mathrm{d}T}{\mathrm{d}p}$$

Proof: By derivative definition,

$$\begin{aligned}
\boldsymbol{G}_{,p}(t) &= \lim_{\Delta p \to 0}\frac{\boldsymbol{G}(t, p+\Delta p) - \boldsymbol{G}(t, p)}{\Delta p} \\
&= \lim_{\Delta p \to 0}\frac{\boldsymbol{G}[t + T(p+\Delta p), p+\Delta p] - \boldsymbol{G}[t+T(p), p]}{\Delta p} \\
&= \lim_{\Delta p \to 0}\frac{\boldsymbol{G}[t + T(p+\Delta p), p+\Delta p] - \boldsymbol{G}[t+T(p), p+\Delta p]}{\Delta p} \\
&\quad + \lim_{\Delta p \to 0}\frac{\boldsymbol{G}[t + T(p), p+\Delta p] - \boldsymbol{G}[t+T(p), p]}{\Delta p} \\
&= \dot{\boldsymbol{G}}(t)T_{,p} + \boldsymbol{G}_{,p}(t+T)
\end{aligned}$$

By the theorem, we have

$$\boldsymbol{U}_{,p}(t) = \dot{\boldsymbol{U}}(t)T_{,p} + \boldsymbol{U}_{,p}(t+T) \tag{26}$$

where $\dot{\boldsymbol{U}}(t) \neq 0$. [If $\dot{\boldsymbol{U}}(t) = 0$, it follows that $\boldsymbol{A} \equiv \boldsymbol{K}$, and then the system will become the one with a constant coefficient. However, we consider the systems with periodic coefficient $\boldsymbol{A}(t)$ here.] Therefore, $\boldsymbol{U}_{,p}(t)$ is periodic (with period T), and so is $\boldsymbol{H}[h_{jk}]$ if and only if $T_{,p} = 0$.

Substituting Eq. (26) into Eq. (22) gives

$$\boldsymbol{H}(t) = \boldsymbol{V}^T(t)[\dot{\boldsymbol{U}}(t)T_{,p} + \boldsymbol{U}_{,p}(t+T)] \tag{27}$$

and taking note of

$$\boldsymbol{H}(t+T) = \boldsymbol{V}^T(t)\boldsymbol{U}_{,p}(t+T) \tag{28}$$

hence, one can obtain

$$\boldsymbol{H}(t+T) - \boldsymbol{H}(t) = -\boldsymbol{V}^T(t)\dot{\boldsymbol{U}}(t)T_{,p} \tag{29}$$

Substituting Eq. (14) into Eq. (29) yields

$$\boldsymbol{H}(t+T) - \boldsymbol{H}(t) = -(\boldsymbol{V}^T\boldsymbol{A}\boldsymbol{U} - \Lambda)T_{,p}, \qquad \forall t \in [0,\infty) \tag{30}$$

Consequently,

$$\boldsymbol{H}(T) - \boldsymbol{H}(0) = [\Lambda - \boldsymbol{L}^T\boldsymbol{A}(0)\boldsymbol{R}]T_{,p} \tag{31a}$$

or

$$h_{jk}(T) - h_{jk}(0) = \left[\lambda_k\delta_{jk} - \boldsymbol{l}_j^T\boldsymbol{A}(0)\boldsymbol{r}_k\right]T_{,p} \tag{31b}$$

Letting $j = k$ and substituting the result obtained into Eq. (25), we obtain

$$\lambda_{k,p} = \frac{1}{T}\int_0^T \boldsymbol{v}_k^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}t - \frac{1}{T}\left[\lambda_k - \boldsymbol{l}_k^T\boldsymbol{A}(0)\boldsymbol{r}_k\right]T_{,p} \qquad k = 1,\ldots,n \tag{32}$$

It should be noted that, if $T_{,p} = 0$, i.e., the period is independent of design parameters, then the formulas become exactly the one developed by Lu and Murthy.[2] In practical engineering, however, the period T included in the equations with periodic coefficients are sometimes dependent on the system variables.[5–7] From Eq. (7b), it can be seen that $\lambda_{k,p}$ is nonunique when T is dependent on p. However, the derivative of its real part $\alpha_{k,p}$ is unique, as is $\rho_{k,p}$.

From Eqs. (7) and (32), it follows that

$$\lambda_{k,p} = \left(\alpha_k + i\omega_k^*\right)_{,p} + (i2m\pi/T)_{,p} = \lambda_{k,p}^* - i(2m\pi/T^2)T_{,p} \qquad k = 1,\ldots,n \tag{33}$$

The derivatives of the principal eigenexponents λ_k^* are

$$\lambda_{k,p}^* = \frac{1}{T}\int_0^T \boldsymbol{v}_k^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}t - \frac{1}{T}\left[\lambda_k^* - \boldsymbol{l}_k^T\boldsymbol{A}(0)\boldsymbol{r}_k\right]T_{,p}$$

Sensitivity of the Periodic Modal Matrix

To obtain $\boldsymbol{U}(t)_{,p}$ from Eq. (23), one has to know $\boldsymbol{H}[h_{jk}]$.

Letting $j \neq k$ in Eq. (21b) yields

$$\dot{h}_{jk} = \boldsymbol{v}_j^T\boldsymbol{A}_{,p}\boldsymbol{u}_k + (\lambda_j - \lambda_k)h_{jk}, \qquad j \neq k, \qquad j,k = 1,\ldots,n \tag{34}$$

The general solution of Eq. (34) can be written as

$$h_{jk}(t) = e^{(\lambda_j-\lambda_k)t}\left[\int\left(\boldsymbol{v}_j^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\right)e^{(\lambda_k-\lambda_j)t}\,\mathrm{d}t + \tilde{c}\right] \tag{35}$$

The following new procedure is suggested to determine the constant $\tilde{c}$ in Eq. (35).

Averaging Eq. (34) over one period yields

$$\left(\frac{1}{T}\right)[h_{jk}(T) - h_{jk}(0)] = (\lambda_j - \lambda_k)\tilde{h}_{jk} + \left(\frac{1}{T}\right)\int_0^T \boldsymbol{v}_j^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}t \tag{36}$$

where

$$\tilde{h}_{jk} \equiv \left(\frac{1}{T}\right)\int_0^T h_{jk}(t)\,\mathrm{d}t$$

Substituting Eq. (31b) into Eq. (36) gives

$$\tilde{h}_{jk} = \frac{1}{(\lambda_k - \lambda_j)T}\left[\int_0^T \boldsymbol{v}_j^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}t + \boldsymbol{l}_j^T\boldsymbol{A}(0)\boldsymbol{r}_kT_{,p}\right], \qquad j \neq k \tag{37}$$

Averaging Eq. (35) over one period and equating the result acquired to Eq. (37), then the constant $\tilde{c}$ can be uniquely determined and thereby the whole off-diagonal elements of $\boldsymbol{H}[h_{jk}]$ are uniquely determined.

To calculate the diagonal elements of $\boldsymbol{H}[h_{jk}]$, because the condition in Eq. (9b) does not uniquely determine the columns in R, we can impose a suitable normalized condition as follows:

$$\boldsymbol{r}_k^T\boldsymbol{r}_k = 1, \qquad k = 1,\ldots,n \tag{38}$$

Differentiating Eqs. (38) and (13) with respect to p, we have

$$\boldsymbol{r}_k^T\boldsymbol{r}_{k,p} = 0 \tag{39}$$

$$\boldsymbol{U}_{,p}(t) = \boldsymbol{P}_{,p}(t)\boldsymbol{R} + \boldsymbol{P}(t)\boldsymbol{R}_{,p} \tag{40}$$

Letting $t = 0$ in Eqs. (13) and (40), we obtain

$$\boldsymbol{U}(0) = \boldsymbol{R}, \qquad \boldsymbol{U}_{,p}(0) = \boldsymbol{R}_{,p} \tag{41}$$

Using Eqs. (23) and (41) written for $t = 0$, we get

$$\boldsymbol{R}_{,p} = \boldsymbol{R}\boldsymbol{H}(0) \tag{42a}$$

or

$$\left[\boldsymbol{r}_{1,p} \;\vdots\; \boldsymbol{r}_{2,p} \;\vdots\; \cdots \;\vdots\; \boldsymbol{r}_{n,p}\right] = \left[\boldsymbol{r}_1 \;\vdots\; \boldsymbol{r}_2 \;\vdots\; \cdots \;\vdots\; \boldsymbol{r}_n\right][h_{jk}(0)] \tag{42b}$$

Using the multiplication of partitioned matrices gives

$$\boldsymbol{r}_{k,p} = \sum_{j=1}^{n}\boldsymbol{r}_jh_{jk}(0) \tag{43}$$

Substituting Eq. (43) into Eq. (39) yields

$$\sum_{j=1}^{n}\boldsymbol{r}_k^T\boldsymbol{r}_jh_{jk}(0) = 0 \tag{44}$$

Utilizing Eq. (38) we get

$$h_{kk}(0) = -\sum_{j\neq k}\boldsymbol{r}_k^T\boldsymbol{r}_jh_{jk}(0), \qquad k = 1,\ldots,n \tag{45}$$

Thus, the whole $h_{kk}(0)$, $k = 1,\ldots,n$ have been determined uniquely.

From Eq. (24) we obtain

$$h_{kk}(t) = \int_0^t \boldsymbol{v}_k^T\boldsymbol{A}_{,p}\boldsymbol{u}_k\,\mathrm{d}\varphi - t\lambda_{k,p} + h_{kk}(0) \tag{46}$$

Thereby, the whole elements of $\boldsymbol{H}[h_{jk}]$ have been acquired. Finally, the derivative of the periodic modal matrix with respect to a system parameter p follows from Eq. (23).

Example

A single example is used to validate the present generalization, of which the closed-form analytical solution can be found.

Consider a linear and periodic system governed by Eq. (1), where

$$\boldsymbol{A}(t) = \begin{bmatrix} a + i(a+b)\cos 2at & -ab + ib(a+b)\sin 2at \\ \dfrac{a}{b} + i\dfrac{a+b}{b}\sin 2at & a - i(a+b)\cos 2at \end{bmatrix}$$

and $T = \pi/a$, and a and b are the system parameters. Therefore, the period T is dependent on the design variable a. The solution of this system can be acquired as

$$\boldsymbol{X}(t) = \begin{bmatrix} \cos at \cdot e^{iat} & -b\sin at \cdot e^{-iat} \\ \dfrac{1}{b}\sin at \cdot e^{iat} & \cos at \cdot e^{-iat} \end{bmatrix}\begin{bmatrix} e^{(a+ib)t} & \\ & e^{(a-ib)t} \end{bmatrix}$$

Table 1 Derivatives of eigenexponents

Eigenexponent	Analytic solution	Present method	Lu and Murthy's method[2]
$\lambda_{,a}$	1	1	$1 \pm i$
$\lambda_{,b}$	$\pm i$	$\pm i$	$\pm i$

Obviously, the principal eigenexponents and the eigenvector complete sets of $\boldsymbol{K}$ are

$$\lambda = a \pm ib, \qquad \boldsymbol{R} = \boldsymbol{L} = \boldsymbol{I}$$

Hence,

$$\boldsymbol{V}^T(t) = \begin{bmatrix} \cos at \cdot e^{-iat} & b \sin at \cdot e^{-iat} \\ -\dfrac{1}{b}\sin at \cdot e^{iat} & \cos at \cdot e^{iat} \end{bmatrix}$$

$$\boldsymbol{U}(t) = \begin{bmatrix} \cos at \cdot e^{iat} & -b \sin at \cdot e^{-iat} \\ \dfrac{1}{b}\sin at \cdot e^{iat} & \cos at \cdot e^{-iat} \end{bmatrix}$$

By calculation, we have

$$\frac{1}{T}\int_0^T \boldsymbol{v}_1^T \boldsymbol{A}_{,a}\boldsymbol{u}_1 \, \mathrm{d}t = 1 + i, \qquad -\frac{1}{T}\left[\lambda_1 - \boldsymbol{l}_1^T \boldsymbol{A}(0)\boldsymbol{r}_1\right] T_{,a} = -i$$

$$\frac{1}{T}\int_0^T \boldsymbol{v}_2^T \boldsymbol{A}_{,a}\boldsymbol{u}_2 \, \mathrm{d}t = 1 - i, \qquad -\frac{1}{T}\left[\lambda_2 - \boldsymbol{l}_2^T \boldsymbol{A}(0)\boldsymbol{r}_2\right] T_{,a} = i$$

$$\frac{1}{T}\int_0^T \boldsymbol{v}_1^T \boldsymbol{A}_{,b}\boldsymbol{u}_1 \, \mathrm{d}t = i, \qquad -\frac{1}{T}\left[\lambda_1 - \boldsymbol{l}_1^T \boldsymbol{A}(0)\boldsymbol{r}_1\right] T_{,b} = 0$$

$$\frac{1}{T}\int_0^T \boldsymbol{v}_2^T \boldsymbol{A}_{,b}\boldsymbol{u}_2 \, \mathrm{d}t = -i, \qquad -\frac{1}{T}\left[\lambda_2 - \boldsymbol{l}_2^T \boldsymbol{A}(0)\boldsymbol{r}_2\right] T_{,b} = 0$$

Thus, we obtain Table 1.

Obviously, when the period T depends on the parameter to be investigated, Lu and Murthy's method for computing the derivatives of eigenexponents is not applicable. As for the derivatives of the periodic modal matrices, by making use of the analytical method and the present formulas, respectively, we would obtain the same results, as follows:

$$\boldsymbol{H}_a = \begin{bmatrix} it & -tbe^{-i2at} \\ \dfrac{t}{b}e^{i2at} & -it \end{bmatrix}$$

$$\boldsymbol{U}_{,a} = \begin{bmatrix} (-t \sin at + it \cos at)e^{iat} & (-tb \cos at + itb \sin at)e^{-iat} \\ \left(\dfrac{t}{b}\cos at + i\dfrac{t}{b}\sin at\right)e^{iat} & (-t \sin at - it \cos at)e^{-iat} \end{bmatrix}$$

$$\boldsymbol{H}_b = \begin{bmatrix} -\dfrac{1}{b}\sin^2 at & -\sin at \cos at \cdot e^{-i2at} \\ -\dfrac{1}{b^2}\sin at \cos at \cdot e^{i2at} & \dfrac{1}{b}\sin^2 at \end{bmatrix}$$

$$\boldsymbol{U}_{,b} = \begin{bmatrix} 0 & -\sin at \cdot e^{-iat} \\ -\dfrac{1}{b^2}\sin at \cdot e^{iat} & 0 \end{bmatrix}$$

According to Eq. (22), where $\boldsymbol{H}_b = \boldsymbol{V}^T\boldsymbol{U}_{,b}$,

$$\boldsymbol{H}_a = \boldsymbol{V}^T\boldsymbol{U}_{,a}$$

Conclusions

This Note established a key theorem to show that the derivative of a periodic function with respect to the system parameter is generally no longer a periodic function when the period T is dependent on this parameter. A new procedure used to determine the constant $\tilde{c}$ in Eq. (35) has been suggested; it substituted for the periodicity condition given by Eq. (27b) in Ref. 2 during $T_{,p} \neq 0$.

The present generalization preserved all of the advantages of the direct analytical approach, and its effectiveness was demonstrated by an example.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant 19472077.

References

[1] Lim, J. W., and Chopra, I., "Stability Sensitivity Analysis of a Helicopter Rotor," *AIAA Journal*, Vol. 28, No. 6, 1990, pp. 1089–1097.

[2] Lu, Y., and Murthy, V. R., "Sensitivity Analysis of Discrete Periodic Systems with Applications to Helicopter Rotor Dynamics," *AIAA Journal*, Vol. 30, No. 8, 1992, pp. 1962–1969.

[3] Yakubovitch, V. A., and Starzhinskii, V. M., *Linear Differential Equations with Periodic Coefficients*, Vols. 1 and 2, Wiley, New York, 1975.

[4] Dugundji, J., and Wendell, J. H., "Some Analysis Methods for Rotating Systems with Periodic Coefficients," *AIAA Journal*, Vol. 21, No. 6, 1982, pp. 890–897.

[5] Perret-Liaudet, J., "An Original Method for Computing the Response of a Parametrically Excited Forced System," *Journal of Sound and Vibration*, Vol. 196, No. 2, 1996, pp. 165–177.

[6] Erdos, G., and Singh, T., "Stability of a Parametrically Excited Damped Invested Pendulum," *Journal of Sound and Vibration*, Vol. 198, No. 5, 1996, pp. 643–650.

[7] Fung, R. F., Huang, J. S., and Chen, W. H., "Dynamic Stability of a Viscoelastic Beam Subjected to Harmonic and Parametric Excitations Simultaneously," *Journal of Sound and Vibration*, Vol. 198, No. 1, 1996, pp. 1–16.

A. Berman
Associate Editor

Free-Edge Interlaminar Stress Analysis of Composite Laminates by Extended Kantorovich Method

Maenghyo Cho*
Inha University, Inchon 402-751, Republic of Korea
and
Jin-Young Yoon†
Agency for Defense Development, Taejon 305-600, Republic of Korea

I. Introduction

BECAUSE of the mismatch of the elastic properties between plies, serious stress concentration/singularity happens near the free edges of composite laminates, requiring fully three-dimensional stress field modeling. The interlaminar stresses at free edges may cause delamination and laminate failure. Thus, the determination of these stresses is an important issue in the strength analysis and design of laminates.

Exact elasticity solutions for interlaminar stresses with free edges do not exist, and researchers have used analytical and numerical methods. However, accurate and efficient methods that satisfy all of the boundary and interface continuity conditions are rare.

After Spilker and Chou[1] demonstrated the importance of satisfying the traction-free conditions at the edges, stress-based methods were proposed. These methods divide stress functions into in-plane and out-of-plane functions. With appropriate stress function

Received Aug. 20, 1998; revision received Nov. 1, 1998; accepted for publication Dec. 27, 1998.

*Assistant Professor, Department of Aerospace Engineering, 253 Yong-Hyun Dong, Nam-ku. Member AIAA.

†Researcher, 4th R&D Center, Yuseong P.O. Box 35-4.